Indian Statistical Institute,Bangalore Centre
B.Math.(Hons.)II Year - 2016-17, First Semester
Optimization

Final Exam — 17 November 2016, 2 pm -5 pm
Instructor: P.S.Datti — Max.Marks: 50

NOTE:

(i) Answer *any* 5 questions. WRITE NEATLY.

(ii) $M_n(\mathbb{R})$ denotes the set of real $n \times n$ matrices.

1. (a) Let $\mathbf{A} \in M_n(\mathbb{R})$ be invertible, $\mathbf{u}, \mathbf{v}$ be column vectors in $\mathbb{R}^n$ and α be a real number. If $\mathbf{A} + \alpha\mathbf{u}\mathbf{v}^t$ is invertible, show, by direct verification, that
$$(\mathbf{A} + \alpha\mathbf{u}\mathbf{v}^t)^{-1} = \mathbf{A}^{-1} + \beta\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^t\mathbf{A}^{-1},$$
for appropriate β. Hence, determine all the values of α for which $\mathbf{A} + \alpha\mathbf{u}\mathbf{v}^t$ is invertible. **(5)**

   (b) Let $\mathbf{e}_j$, $1 \le j \le n$, denote the standard unit vectors in $\mathbb{R}^n$. Put $\mathbf{E}_{ij} = \mathbf{e}_i\mathbf{e}_j^t$ for $1 \le i, j \le n$. By using (a) above or otherwise, find all real λ and μ such that the matrix $\mathbf{I} + \lambda\mathbf{E}_{1n} + \mu\mathbf{E}_{n1}$ is invertible. Find an expression for the inverse in that case. **(5)**

2. Using the singular value decomposition or otherwise, prove the following:

   (a) Suppose $\mathbf{A}, \mathbf{B}$ are real $m \times n$ matrices such that $\mathbf{A}^t\mathbf{A} = \mathbf{B}^t\mathbf{B}$. Show that there is an orthogonal $m \times m$ matrix $\mathbf{U}$ such that $\mathbf{A} = \mathbf{U}\mathbf{B}$. **(5)**

   (b) Suppose $\mathbf{A} \in M_n(\mathbb{R})$. Show that the eigenvalues of $\mathbf{A}^t\mathbf{A}$ and $\mathbf{A}\mathbf{A}^t$ are the same and with the same algebraic multiplicities. **(5)**

3. (a) Let $\mathbf{A} = ((a_{ij})) \in M_n(\mathbb{R})$ be a positive matrix, that is $a_{ij} > 0$ for all i, j. Suppose there is a $\lambda > 0$ and a vector $\mathbf{x} \ge \mathbf{0}$, $\mathbf{x} \ne \mathbf{0}$ such that $\mathbf{A}\mathbf{x} \ge \lambda\mathbf{x}$, $\mathbf{A}\mathbf{x} \ne \lambda\mathbf{x}$. Show that there is a vector $\mathbf{y} \ge \mathbf{0}$, $\mathbf{y} \ne \mathbf{0}$ such that $\mathbf{A}\mathbf{y} > \lambda\mathbf{y}$. **(5)**

   (b) Suppose $\mathbf{A} \in M_n(\mathbb{R})$ is a non-negative, irreducible matrix. Let $\rho = \rho_{\mathbf{A}}$ be the the eigenvalue of $\mathbf{A}$ such that ρ equals the spectral radius of $\mathbf{A}$. If $\mathbf{x}$ is a non-zero real or complex $n-$vector such that $(\mathbf{A} - \rho\mathbf{I})^2\mathbf{x} = \mathbf{0}$, show that $(\mathbf{A} - \rho\mathbf{I})\mathbf{x} = \mathbf{0}$. **(5)**

4. (a) Show that 1 is the dominant eigenvalue of the doubly stochastic matrix $\mathbf{A} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$ and hence find the limit $\lim_{k\to\infty} \mathbf{A}^k$. **(5)**

(b) Suppose $\mathbf{A} \in M_n(\mathbb{R})$ is a non-negative, irreducible matrix and $\mathbf{B} \in M_n(\mathbb{R})$ is a non-negative matrix such that $\mathbf{A} - \mathbf{B}$ is also a non-negative matrix. If $r(\mathbf{A})$ and $r(\mathbf{B})$ denote the spectral radii of $\mathbf{A}$ and $\mathbf{B}$, respectively, show that $r(\mathbf{B}) \leq r(\mathbf{A})$, with equality holding only if $\mathbf{A} = \mathbf{B}$. **(5)**

5. (a) Solve the following using simplex method:

$$\begin{aligned}
&\text{minimize } 5x_1 - 8x_2 - 3x_3 \\
&\text{subject to } 2x_1 + 5x_2 - x_3 \leq 1 \\
&\qquad -3x_1 - 8x_2 + 2x_3 \leq 4 \\
&\qquad -2x_1 - 12x_2 + 3x_3 \leq 9 \\
&\qquad\quad x_i \geq 0,\ i = 1, 2, 3.
\end{aligned}$$

(5)

(b) Find the dual of the linear programming:

$$\text{maximize } \mathbf{c}^t\mathbf{x}, \text{ subject to } \ \mathbf{A}\mathbf{x} \leq \mathbf{b},$$

where, $\mathbf{A}$ is an $m \times n$ real matrix and $\mathbf{b}, \mathbf{c}$ are given column vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. In this set up, state and prove the weak duality lemma. **(5)**

6. (a) Let $\mathbf{A}$ be an $m \times n$ real matrix, whose rank is m; $\mathbf{b}, \mathbf{c}$ are given column vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Consider the following linear programming:

$$\text{minimize } \mathbf{c}^t\mathbf{x}, \text{ subject to } \ \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}.$$

If $\mathbf{A} = (\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n)$, assume that the matrix $\mathbf{B} = (\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_m)$ is non-singular and that there is a vector $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{x} \geq \mathbf{0}$ satisfying $\mathbf{B}\mathbf{x} = \mathbf{b}$.

For $\varepsilon > 0$ small, consider the system $\mathbf{A}\mathbf{x} = \mathbf{b}(\varepsilon)$, where $\mathbf{b}(\varepsilon) = \mathbf{b} + \varepsilon\mathbf{a}_1 + \cdots + \varepsilon^n\mathbf{a}_n$. Show that there is a vector $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{y} > \mathbf{0}$ satisfying $\mathbf{B}\mathbf{y} = \mathbf{b}(\varepsilon)$, for some range of $\varepsilon > 0$. **(5)**

(b) Consider the linear programme (P) of the form

$$\begin{aligned}
&\text{minimize } \mathbf{q}^t\mathbf{z} \\
&\text{subject to } \mathbf{M}\mathbf{z} \geq -\mathbf{q} \\
&\qquad\qquad\quad \mathbf{z} \geq \mathbf{0},
\end{aligned}$$

where $\mathbf{M} \in M_k(\mathbb{R})$ is a skew-symmetric matrix, that is, $\mathbf{M} = -\mathbf{M}^t$ and $\mathbf{q} \in \mathbb{R}^k$. Show that the problem (P) and its dual are the same. Further, show that any feasible solution of (P) is also optimal. **(5)**

7. (a) Suppose C is a convex set in $\mathbb{R}^n$ and $k \geq 2$. Let $\mathbf{x}_1, \dots, \mathbf{x}_k$ are in C and $t_1, \dots, t_k$ are non-negative real numbers such that $t_1 + \dots + t_k = 1$. Show that $t_1\mathbf{x}_1 + \dots + t_k\mathbf{x}_k$ is also in C. **(2)**

   (b) Define an extreme point of a convex set in $\mathbb{R}^n$. **(1)**

   (c) Consider the set of constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, where $\mathbf{A}$ is a real $m \times n$ matrix of rank m and $\mathbf{b} \in \mathbb{R}^m$.

      i. Define a basic feasible solution of the above set of constraints. **(1)**

      ii. State and prove the result concerning the extreme points of the set of all feasible solutions of the above set of constraints. **(2 + 4)**